Pulse Width Modulation in Sampled Data Systems

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Posted on http://mviordache.name in August 2019.

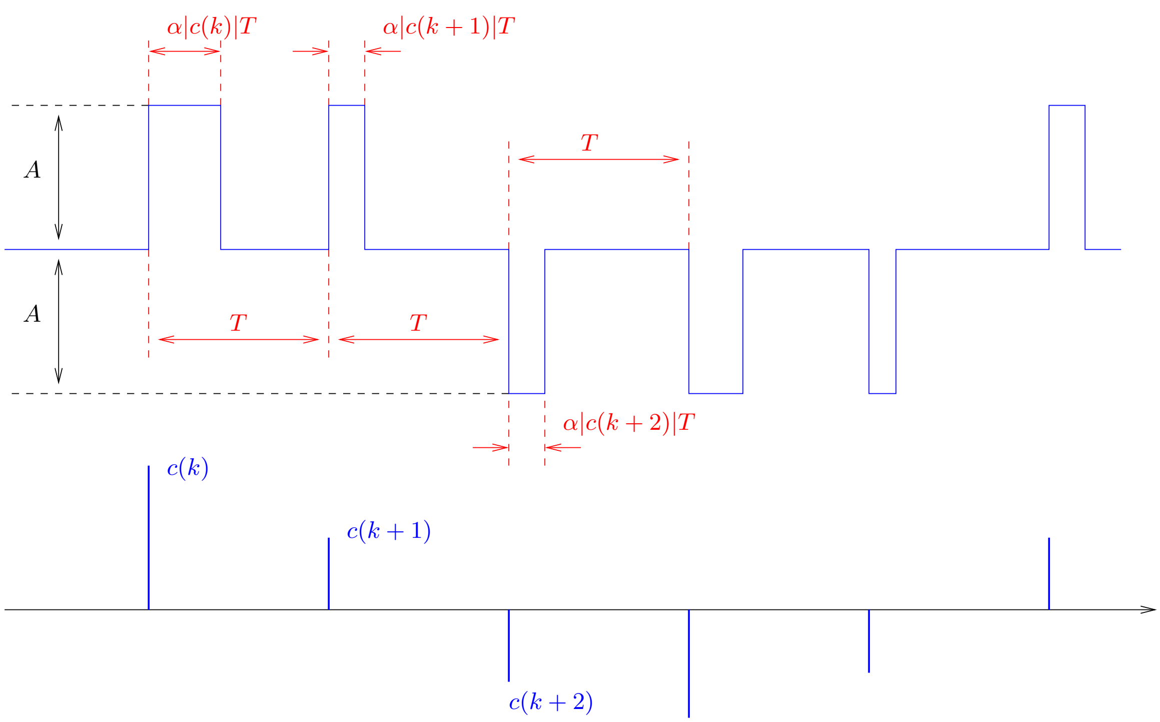
And unto man he said, Behold, the fear of the Lord, that is wisdom; and to depart from evil is understanding. Job 28:28

Pulse-width modulation (PWM) is common in control systems involving power electronic devices. While the precise analysis of control systems that use PWM is rather complex, an approximate solution is much simpler and can be easily incorporated in an undergraduate control-systems class. This document introduces a simple approximation method, defines an approximation error, and shows that the error converges to zero as the PWM period goes to zero. Some remarks on the applicability of this approximation are included at the end of the paper.

Consider a PWM module that converts a discrete-time signal $c(k)$ to a three-level PWM signal $p(t)$ as follows:

$$p(t) = \begin{cases} A & \text{if } c(k) \geq 0 \text{ and } kT \leq t < kT + \alpha|c(k)|T \\ -A & \text{if } c(k) < 0 \text{ and } kT \leq t < kT + \alpha|c(k)|T \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here, A denotes the pulse size, α is a constant, and T is the cycle time (period) of the PWM signal. Note that the duty cycle is $\alpha|c(k)|$. An illustration of the PWM signal $p(t)$ and the corresponding discrete-time signal $c(k)$ is shown in the figure.



Note that the PWM module is similar to a data hold in that it converts a discrete-time signal $c(k)$ to a continuous-time signal $p(t)$. A practical example of a three-level PWM signal $p(t)$ would be the voltage applied to a brushed DC motor driven by an H-bridge driver, while $c(k)$ would represent the control signal calculated by a computer or a microcontroller.

In the context of PWM, a standard approximation technique is averaging [2], [1]. In this approximation, $p(t) \approx p_1(t)$, where p_1 is a piece-wise constant signal representing the average of $p(t)$ over each PWM cycle:

$$p_1(t) = A \cdot \alpha \cdot c(k), \text{ for } kT \leq t < (k+1)T. \tag{2}$$

This approximation makes the PWM module similar to a zero-order data hold, simplifying considerably the analysis of the control system. In the frequency domain, the Laplace transform $P_1(s)$ of $p_1(t)$ is,

$$P_1(s) = \sum_{k=0}^{\infty} A \cdot \alpha \cdot c(k) \cdot \frac{1 - e^{-sT}}{s} \cdot e^{-skT}. \tag{3}$$

With the substitution $z = e^{-sT}$,

$$P_1(s) = A \cdot \alpha \cdot C(z) \cdot \frac{1 - e^{-sT}}{s}. \tag{4}$$

A first-order approximation $e^{-sT} \approx 1 - sT$ is reasonable when $|s|T \ll 1$. If this approximation is applied to (4), we obtain that $P_1(s) \approx P_2(s)$, where

$$P_2(s) = A \cdot T \cdot \alpha \cdot C(z). \tag{5}$$

Note that the inverse Laplace transform of $P_2(s)$ is $p_2(t) = AT\alpha \sum_{k=0}^{\infty} c_k \delta(t - kT)$, where $\delta(t)$ is the delta Dirac "function". Clearly, the original PWM signal $p(t)$ is quite different from $p_2(t)$. Nonetheless, it can be shown that as $T \to 0$, the two signals have the same effect on a stable continuous-time system. In this sense, $p_2(t)$ approximates $p(t)$.

The approximation of $p(t)$ by $p_2(t)$ can be justified as follows. Assume that the PWM signal $p(t)$ is applied to a plant of transfer function $G(s)$. The following assumptions will be made about the plant:

(a) The impulse response $g(t)$ of the plant is defined for all $t \geq 0$ (it contains no δ function terms).

(b) $g(t)$ is continuous and differentiable for all $t > 0$.

(c) $g'(t)$ is continuous for all $t > 0$.

(d) There are $M > 0$ and $\beta > 0$ so that $|g'(t)| \leq Me^{-\beta t}$ for all $t \geq 0$.

The assumptions are satisfied by all rational transfer functions $G(s)$ that are stable and have the degree of the numerator less than the degree of the denominator. Note that the assumptions allow a discontinuity of $g(t)$ at $t = 0$. For example, if $G(s) = \frac{1}{s+4}$, there is a discontinuity at time $t = 0$.

Note that the PWM signals considered here cannot represent an unbounded modulating signal $c(k)$. In fact, equation (1) implicitly assumes $\alpha|c(k)| < 1$. This assumption is stated explicitly here.

(e) For all $k \geq 0$, $|c(k)| < \frac{1}{\alpha}$.

The output of the plant is $Y(s) = G(s)P(s)$, where $P(s)$ can be calculated from (1) as

$$P(s) = \frac{A}{s} \sum_{k=0}^{\infty} \text{sgn}(c(k)) e^{-skT} \left(1 - e^{-s\alpha|c(k)|T}\right) \tag{6}$$

Let $g(t)$ and $h(t)$ be the inverse Laplace transforms of $G(s)$ and $G(s)/s$, respectively. We obtain that

$$y(t) = A\sum_{k=0}^{\infty} \operatorname{sgn}(c(k))\left(h(t-kT) - h(t-kT-\tau_k)\right), \tag{7}$$

where $\tau_k = \alpha|c(k)|T$. Since $h(t) = \int_0^t g(x)dx$ for $t \geq 0$, it follows that h is continuous. Moreover, $h' = g$ is also continuous for $t \neq 0$. Let $m \geq 0$ be an integer so that $mT \leq t < (m+1)T$. Since $h(t) = 0$ for $t < 0$, $h(t-kT) = h(t-kT-\tau_k) = 0$ for all $k > m$. Therefore, the summation of (7) can be done from $k = 0$ to $k = m$. If $k < m$, then $h(t-kT) - h(t-kT-\tau_k) = g(t-kT-x_k)\tau_k$ for some x_k satisfying $0 \leq x_k \leq \tau_k$. Note that x_k is not a constant, but depends on t. The expression of $y(t)$ can be simplified to

$$y(t) = Ad_m + AT\alpha\sum_{k=0}^{m-1} c(k)g(t-kT-x_k), \tag{8}$$

where $d_m = \operatorname{sgn}(c(m))\left(h(t-mT) - h(t-mT-\tau_m)\right)$. If $t \geq mT+\tau_m$, then $d_m = T\alpha c(m)g(t-mT-x_m)$. If $t < mT+\tau_m$, then $d_m = \operatorname{sgn}(c(m))h(t-mT)$, which implies $d_m = \operatorname{sgn}(c(m))g(t-mT-x_m)(t-mT)$ for some x_m in the range $0 \leq x_m \leq t-mT$. The last equation is more conveniently written as $d_m = T\alpha c(m)g(t-mT-x_m)(t-mT)/\tau_m$. Summarizing:

$$d_m = \begin{cases} T\alpha c(m)g(t-mT-x_m) & \text{if } t \geq mT+\tau_m \\ T\alpha c(m)g(t-mT-x_m)(t-mT)/\tau_m & \text{otherwise.} \end{cases} \tag{9}$$

Since m was defined so that $mT \leq t < (m+1)T$, it can be concluded that

$$|d_m| \leq T\alpha|c(m)g(t-mT-x_m)|. \tag{10}$$

If $P_2(s)$ is applied to $G(s)$, the output $y_2(t)$ can be calculated from (5) as

$$y_2(t) = AT\alpha\sum_{k=0}^{m} c(k)g(t-kT). \tag{11}$$

In the equation above, the summation is from $k = 0$ to $k = m$ because $g(t) = 0$ for $t < 0$, and $mT \leq t < (m+1)T$. The approximation error $e(t) = y(t) - y_2(t)$ is

$$e(t) = Ad_m - AT\alpha c(m)g(t-mT) + AT\alpha\sum_{k=0}^{m-1} c(k)\left(g(t-kT-x_k) - g(t-kT)\right). \tag{12}$$

Since $g(t)$ and $g'(t)$ were assumed to be continuous for $t > 0$, $g(t-kT-x_k) - g(t-kT) = x_k g'(t-kT-v_k)$ for some v_k in the range $0 \leq v_k \leq x_k$. Since $0 \leq x_k \leq T$, in view of the assumptions $\alpha|c(k)| < 1$ and $|g'(t)| < Me^{-\beta t}$,

$$\left|AT\alpha\sum_{k=0}^{m-1} c(k)\left(g(t-kT-x_k) - g(t-kT)\right)\right| < AT^2M\sum_{k=0}^{m-1} e^{-\beta(t-(k+1)T)}. \tag{13}$$

Since $t \geq mT$, the sum of exponentials is upper bounded by $\frac{1-e^{-\beta mT}}{1-e^{-\beta T}} < \frac{1}{1-e^{-\beta T}}$. Therefore, (13) can be written with a simpler upper bound that is independent of t and m:

$$\left|AT\alpha\sum_{k=0}^{m-1} c(k)\left(g(t-kT-x_k) - g(t-kT)\right)\right| < \frac{AT^2M}{1-e^{-\beta T}}. \tag{14}$$

Upper bounds of the remaining terms of (12) are as follows. The assumptions made on the plant ensure that $g(t)$ is bounded. Let $C = \sup_{t \geq 0} g(t)$. In view of (10),

$$|Ad_m| \leq ATC. \tag{15}$$

The same bound is found for the second term of (12):

$$|AT\alpha c(m)g(t - mT)| \leq ATC. \tag{16}$$

In view of (14), (15), and (16), $\lim_{T \to 0} e(t)$ exists and equals zero:

$$\lim_{T \to 0} e(t) = 0. \tag{17}$$

This result holds true also if the plant has a delay τ. The proof can be extended to this case by using $t > \tau$ ($t \geq \tau$) in the place of $t > 0$ ($t \geq 0$) in the assumptions and in the proof.

As indicated above, as $T \to 0$, it is reasonable to approximate the PWM signal with $P_2(s)$ defined in (5). From a practical standpoint, however, it is important to know how small should be T. Intuitively, the approximation should be reasonable when the continuous-time plant is slow in comparison to the frequency of the PWM signal. Without providing a formal proof, it will be suggested here that the approximation is likely reasonable when $|p|T \ll 1$ for all significant poles p of the plant.

Let R_i denote the *residue* of $Y(s)e^{st}$ at the pole p_i of Y:

$$R_i(t) = \frac{1}{(n_i - 1)!} \lim_{s \to p_i} \frac{d^{(n_i - 1)}}{ds^{(n_i - 1)}} \left[(s - p_i)^{n_i} Y(s) e^{st} \right], \tag{18}$$

where n_i is the multiplicity of the pole p_i. The output $y(t)$ can be calculated as

$$y(t) = \sum_i R_i(t). \tag{19}$$

(Note that the summation is infinite if the number of poles is infinite. For example, $\frac{1}{1 - e^{-sT}}$ has poles at $\frac{2\pi mj}{T}$ for all integers m.) Since R_i is calculated as $s \to p_i$, a first-order approximation $e^{-sT} \approx 1 - sT$ would seem justified only if $|p_i|T \ll 1$. Assuming that the plant $G(s)$ could be regarded as a low-pass filter, a finite number of R_i terms will be significant, while the others could be neglected. The significant terms R_i will correspond to poles p_i that are relatively close to the origin. Assuming $|p_i|T \ll 1$ for every such pole, an approximation $e^{-sT} \approx 1 - sT$ would seem reasonable, since $|s|T \ll 1$ holds true in the context of all significant terms R_i. By applying the approximation $e^{-sT} \approx 1 - sT$ to equation (6), and accounting for $\alpha|c(k)| < 1$, we obtain

$$P(s) \approx \frac{AT}{s} \sum_{k=0}^{\infty} \operatorname{sgn}(c(k)) e^{-skT} \cdot s \cdot \alpha \cdot |c(k)|. \tag{20}$$

In view of (5), the relation above states that $P(s) \approx P_2(s)$.

References

[1] S. Cuk and R.D. Middlebrook. A general unified approach to modelling switching DC-to-DC converters in discontinuous conduction mode. In *1977 IEEE Power Electronics Specialists Conference*, pages 36–57, 1977.

[2] J. Sun. Pulse-width modulation. In F. Vasca and L. Iannelli, editors, *Dynamics and Control of Switched Electronic Systems*, Advances in Industrial Control, pages 25–61. Springer-Verlag, 2012.