

Power in AC Circuits

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The LORD is my strength and song, and is become my salvation. Psalm 118:14

Power analysis of linear AC circuits with sinusoidal excitation is a standard topic of introductory electric-circuits textbooks. This document is meant as a supplement to a textbook chapter on power. Its goal is to provide additional insight into the meaning of the various AC power concepts by means of a time-domain analysis. The relationship between the various power concepts is explained and the main results are summarized.

Instantaneous Power. Consider a two-terminal network or component, as shown in Figure 1(a). The absorbed **instantaneous power** is defined as

$$p(t) = v(t)i(t). \quad (1)$$

The instantaneous power can also be calculated for a network or component with more than two terminals. Consider the component with n terminals shown in Figure 1(b). Let v_1, v_2, \dots, v_n be the voltages of the n terminals. Suppose that v_1, v_2, \dots, v_n are defined with respect to a common reference point, where the reference point is marked by the ground symbol in the figure. The instantaneous power has the formula

$$p(t) = \sum_{k=1}^n v_k(t)i_k(t). \quad (2)$$

Note that if the reference point is changed, the voltages v_1, v_2, \dots, v_n are changed by the same amount. If the voltage of the new reference point is v_0 with respect to the old reference point, then $v_1 \rightarrow v_1 + v_0$, $v_2 \rightarrow v_2 + v_0$, \dots , $v_n \rightarrow v_n + v_0$. However, the result of formula (2) will not change, since Kirchoff's current law implies that $v_0 \sum i_k = 0$. It could be noted also that equation (1) is the same as formula (2) when the reference point is the terminal where $v(t)$ has the $-$ sign.

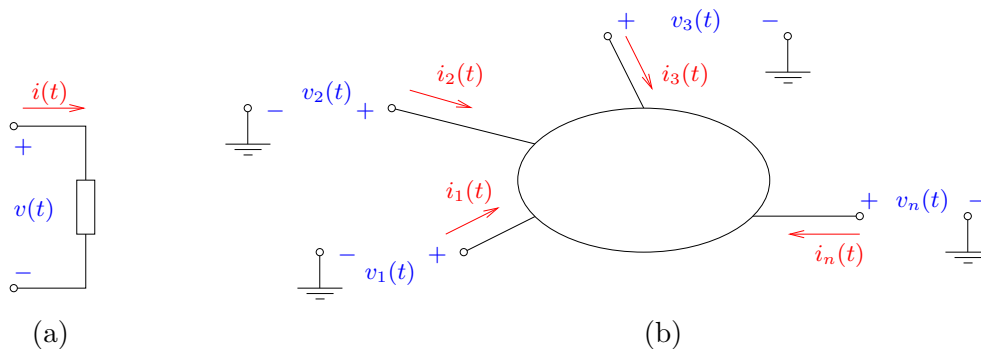


Figure 1: Components with two or more terminals.

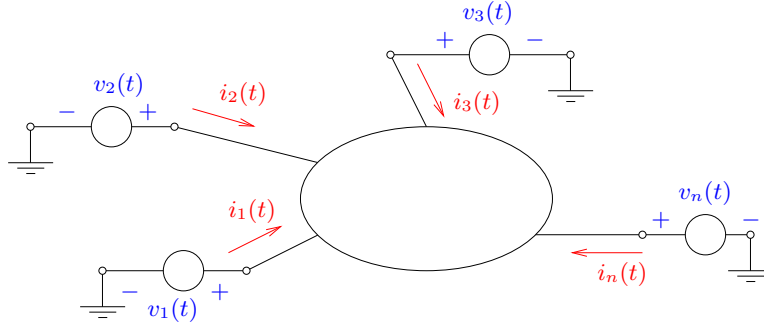


Figure 2: An equivalent representation of the n -terminal component of Figure 1(b).

Formula (2) can be justified by considering the circuit of Figure 2, showing an equivalent representation of the n -terminal component of Figure 1(b). In this equivalent representation, the currents and voltages of the component are caused by n sources of current or voltage. The power absorbed by the component is the same, since the component has the same voltages and currents as in Figure 1(b). Now the power generated by the n sources must equal the power $p(t)$ absorbed by the component. Then, equation (2) is obtained based on the observation that the total power generated by the sources is $\sum_{k=1}^n v_k(t)i_k(t)$.

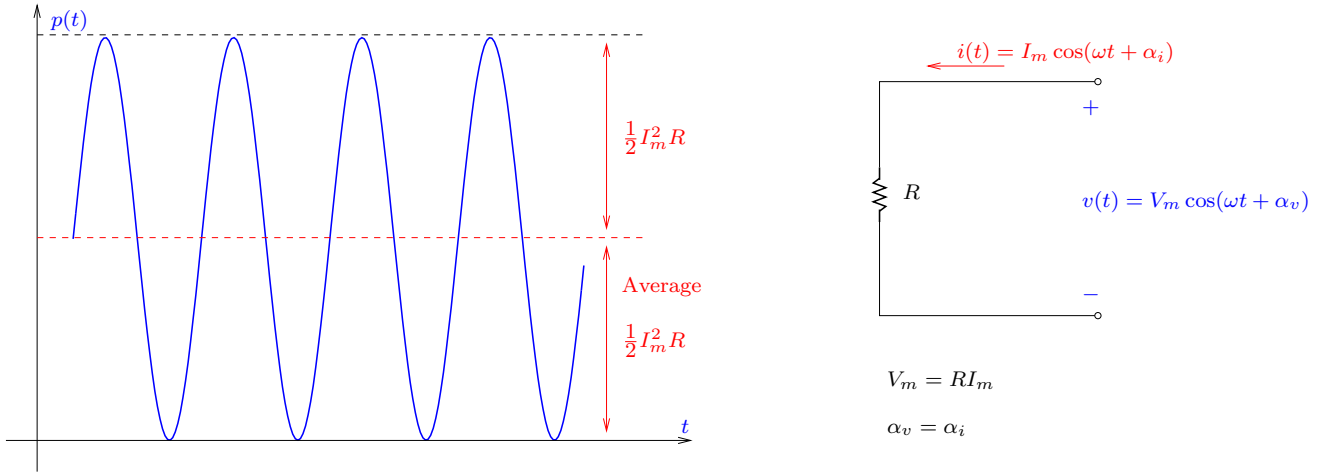


Figure 3: The instantaneous power of a resistor.

Examples. The application of equation (1) to AC circuits will be illustrated on a few examples. First, assume that the two-terminal component of Figure 1(a) is a resistor of value R . In view of Ohm's law, $p(t) = Ri(t)^2$. Moreover, if $i(t) = I_m \cos(\omega t + \alpha_i)$, then

$$p(t) = RI_m^2 \cos^2(\omega t + \alpha_i).$$

Recalling the formula $\cos^2(x) = \frac{1 + \cos(2x)}{2}$, we obtain

$$p(t) = \frac{RI_m^2}{2} + \frac{RI_m^2}{2} \cos(2\omega t + 2\alpha_i). \quad (3)$$

The equation indicates that the instantaneous power oscillates between 0 and RI_m^2 , with an average of $\frac{RI_m^2}{2}$, as shown also in Figure 3.

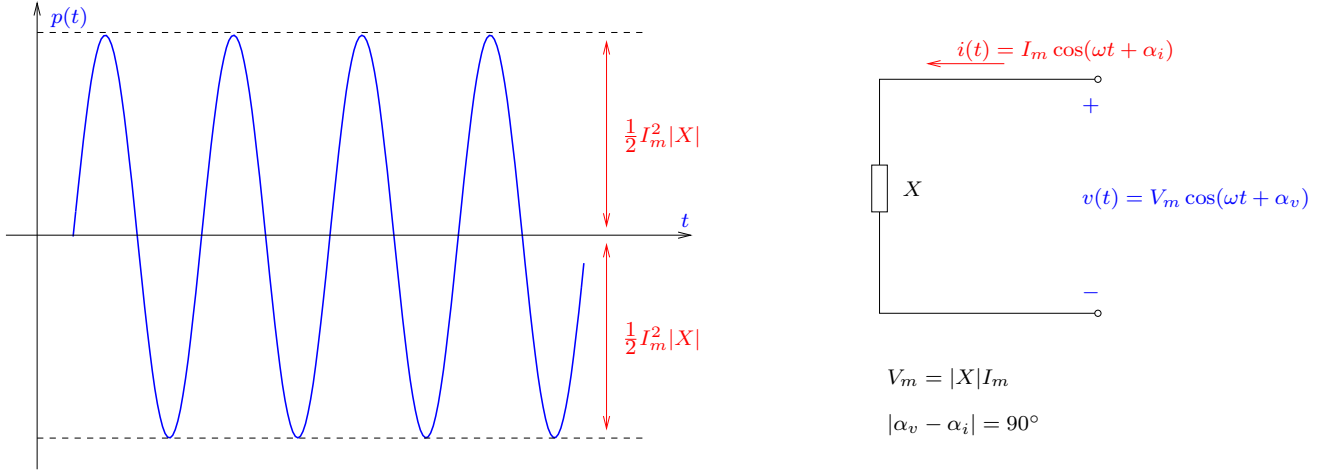


Figure 4: The instantaneous power of a reactive component.

Assume now that the two-terminal component of Figure 1(a) is a capacitor of value C . Since $i(t) = C\dot{v}(t)$, assuming $v(t) = V_m \cos(\omega t + \alpha_v)$,

$$p(t) = -C\omega V_m^2 \cos(\omega t + \alpha_v) \sin(\omega t + \alpha_v).$$

Recalling that $\cos(x) \sin(x) = \frac{\sin(2x)}{2}$,

$$p(t) = -\frac{1}{2}C\omega V_m^2 \sin(2\omega t + 2\alpha_v). \quad (4)$$

The equation indicates that the instantaneous power oscillates between $-\frac{C\omega V_m^2}{2}$ and $\frac{C\omega V_m^2}{2}$ with a zero average. Recalling that the reactance $X = -\frac{1}{C\omega}$ and $V_m = |X|I_m$, we could also say that the instantaneous power oscillates with an amplitude of $\frac{1}{2}I_m^2|X|$, as shown in Figure 4.

In the same way as before, assuming that the two-terminal component of Figure 1(a) is an inductor of value L ,

$$p(t) = -\frac{1}{2}L\omega I_m^2 \sin(2\omega t + 2\alpha_i). \quad (5)$$

The graph of $p(t)$ is shown in Figure 4. As could be seen from equations (4) and (5), the average power of a reactive component (a capacitor or an inductor) is zero.

Average, Apparent, and Reactive Powers. Substituting the voltage $v(t) = V_m \cos(\omega t + \alpha_v)$ and the current $i(t) = I_m \cos(\omega t + \alpha_i)$ in equation (1):

$$p(t) = V_m I_m \cos(\omega t + \alpha_v) \cos(\omega t + \alpha_i). \quad (6)$$

Recalling the formula $\cos(x) \cos(y) = \frac{\cos(x-y) + \cos(x+y)}{2}$,

$$p(t) = \frac{V_m I_m}{2} \cos(\alpha_v - \alpha_i) + \frac{V_m I_m}{2} \cos(2\omega t + \alpha_i + \alpha_v). \quad (7)$$

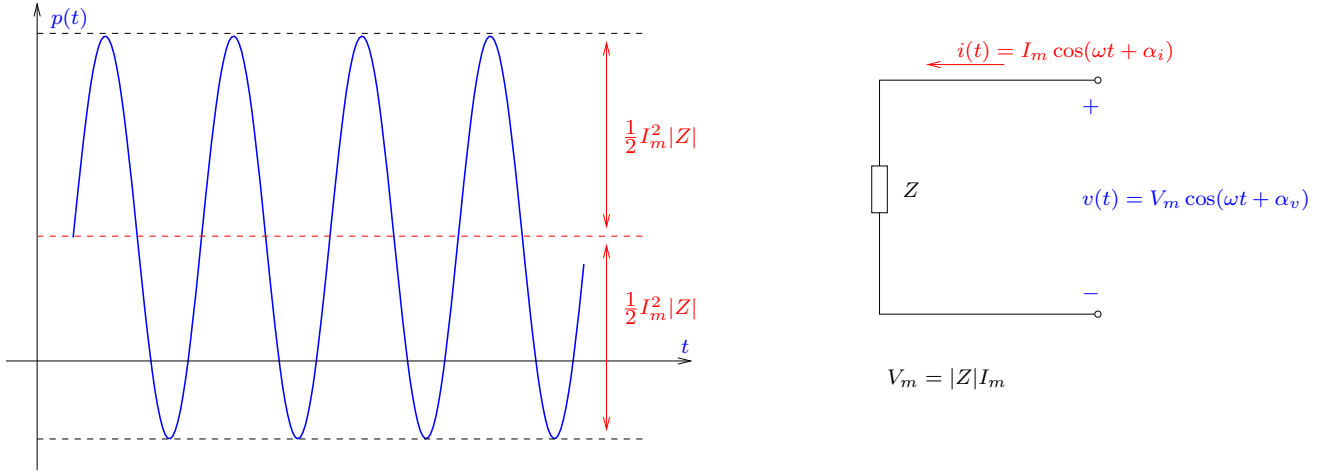


Figure 5: The instantaneous power for a component or network of impedance Z .

The graph of $p(t)$ is shown in Figure 5.

Considering equation (7), the constant term is the average of $p(t)$. Thus, **the average power** is

$$P = \frac{V_m I_m}{2} \cos(\alpha_v - \alpha_i). \quad (8)$$

By definition, the **apparent power** is

$$S = \frac{V_m I_m}{2} \quad (9)$$

In view of equation (7), the instantaneous power can be written as

$$p(t) = P + S \cos(2\omega t + \alpha_i + \alpha_v). \quad (10)$$

Applying the formula $\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$ to the above equation with $x = \alpha_v - \alpha_i$ and $y = 2\omega t + 2\alpha_i$, the instantaneous power can be written in the form

$$p(t) = P + S \cos(\alpha_v - \alpha_i) \cos(2\omega t + 2\alpha_i) - S \sin(\alpha_v - \alpha_i) \sin(2\omega t + 2\alpha_i). \quad (11)$$

In view of equations (8) and (9),

$$P = S \cos(\alpha_v - \alpha_i). \quad (12)$$

Moreover, the amplitude of the last term of equation (11) is known as **reactive power**:

$$Q = S \sin(\alpha_v - \alpha_i). \quad (13)$$

Equation (11) can be written in terms of P and Q as

$$p(t) = P + P \cos(2\omega t + 2\alpha_i) - Q \sin(2\omega t + 2\alpha_i). \quad (14)$$

Physical Interpretation. Equation (14) has a simple physical interpretation. Assume a load represented by a resistor of value R in series with a reactive component of reactance X . The instantaneous power of the load can be decomposed into the sum of the power of the resistor $p_R(t)$ and the power of the reactive component $p_X(t)$:

$$p(t) = p_R(t) + p_X(t). \quad (15)$$

In view of equations (3), (4), and (5), $p_R(t)$ and $p_X(t)$ can be identified with the terms of equation (14) as follows:

$$p_R(t) = P + P \cos(2\omega t + 2\alpha_i) \quad (16)$$

$$p_X(t) = -Q \sin(2\omega t + 2\alpha_i). \quad (17)$$

We can conclude that the reactive power Q is the peak rate at which energy flows to or from the reactive component. We arrive to the same conclusion if the load is a parallel combination of a resistor of conductance G and a reactive component of susceptance B . First, let us note that if we apply to equation (10) the formula $\cos(y - x) = \cos(x)\cos(y) + \sin(x)\sin(y)$ with $x = \alpha_v - \alpha_i$ and $y = 2\omega t + 2\alpha_v$, we obtain:

$$p(t) = P + P \cos(2\omega t + 2\alpha_v) + Q \sin(2\omega t + 2\alpha_v). \quad (18)$$

Denoting by $p_G(t)$ and $p_B(t)$ the instantaneous powers of the resistor and the reactive component, respectively,

$$p(t) = p_G(t) + p_B(t). \quad (19)$$

In view of equations (3), (4), and (5), $p_G(t)$ and $p_B(t)$ can be identified with the terms of equation (18) as follows:

$$p_G(t) = P + P \cos(2\omega t + 2\alpha_v) \quad (20)$$

$$p_B(t) = Q \sin(2\omega t + 2\alpha_v). \quad (21)$$

More on the Reactive Power. The average power P was defined as the average of the instantaneous power $p(t)$. The reactive power also is the average of a function $q(t)$ defined as follows. Given the n -terminal component of Figure 1(b), let

$$q(t) = \sum_{k=1}^n v_k(t) i_k(t + t_0), \quad (22)$$

where $t_0 = \frac{\pi}{2\omega}$. It has been assumed here that all currents and voltages are sinusoidal and have the frequency ω . Note that in the case of the two-terminal component of Figure 1(a), the equation can be simplified to

$$q(t) = v(t) i(t + t_0). \quad (23)$$

Substituting $v(t) = V_m \cos(\omega t + \alpha_v)$ and $i(t) = I_m \cos(\omega t + \alpha_i)$, noticing also that $i(t + t_0) = -I_m \sin(\omega t + \alpha_i)$, we obtain

$$q(t) = -V_m I_m \cos(\omega t + \alpha_v) \sin(\omega t + \alpha_i). \quad (24)$$

Recalling that $\cos(x)\sin(y) = \frac{-\sin(x - y) + \sin(x + y)}{2}$,

$$q(t) = \frac{1}{2} V_m I_m \sin(\alpha_v - \alpha_i) - \frac{1}{2} V_m I_m \sin(2\omega t + \alpha_v + \alpha_i), \quad (25)$$

that is,

$$q(t) = Q - S \sin(2\omega t + \alpha_v + \alpha_i). \quad (26)$$

The equation above shows that Q is the average of $q(t)$.

Total Average Power and Total Reactive Power. The average power of a circuit network can be calculated as the sum of the average powers of the network components. The same is true of the reactive

power. The reactive power of a network is the sum of the reactive powers of the network components. To justify these properties, consider a network of m components. Let $p_k(t)$ denote the instantaneous power of component k , for $k = 1, 2, \dots, m$. The instantaneous power of the network is

$$p(t) = \sum_{k=1}^m p_k(t). \quad (27)$$

Therefore, the average power of the network is

$$P = \sum_{k=1}^m P_k, \quad (28)$$

where P_k is the average power of the component k . The same relationship can be established for the reactive power. Let $q(t)$ be defined as in (22) for the entire network, and let $q_k(t)$ be defined as in (22) for the component k , with $k = 1, 2, \dots, m$. Let's prove that $q(t)$ is the sum of $q_1(t), q_2(t), \dots, q_m(t)$. Let N be the number of nodes of the network and n the number of terminals. Without loss of generality, we will assume that there is no node connected to more than one terminal. Let v_k be the voltage of the node k with respect to the reference. For every node $k = 1, 2, \dots, N$ connected to a terminal, let i_k be the terminal current flowing into the node. For every node k that is not connected to a terminal, let $i_k = 0$. Suppose that the nodes $1, 2, \dots, n$ are the nodes connected to the n terminals. It follows that $i_k = 0$ for $k > n$. This implies that equation (22) is equivalent to

$$q(t) = \sum_{k=1}^N v_k(t) i_k(t + t_0), \quad (29)$$

Let $i_{k,c}$ be the current flowing from the node k to the component c , for $k = 1, 2, \dots, N$ and $c = 1, 2, \dots, m$. If the component c is not connected to the node k , then $i_{k,c} = 0$. By Kirchhoff's current law,

$$q(t) = \sum_{k=1}^N \left(v_k(t) \sum_{c=1}^m i_{k,c}(t + t_0) \right). \quad (30)$$

By changing the summation order,

$$q(t) = \sum_{c=1}^m \sum_{k=1}^N v_k(t) i_{k,c}(t + t_0), \quad (31)$$

which leads to

$$q(t) = \sum_{c=1}^m q_k(t). \quad (32)$$

It follows that

$$Q = \sum_{k=1}^m Q_k, \quad (33)$$

where Q_k is the average of $q_k(t)$, that is, the reactive power of the component k .

Note Table 1 for examples illustrating the calculation of the average and reactive powers.

Relation of $p(t)$ to P , Q , and S . Given an arbitrary network with two terminals, Figure 6 illustrates the relation between the instantaneous power $p(t)$ and the apparent power S , the average power P , and the reactive power Q . The figure considers also the special cases when the network is equivalent with a resistor or with a reactive component. The following remarks could be made:

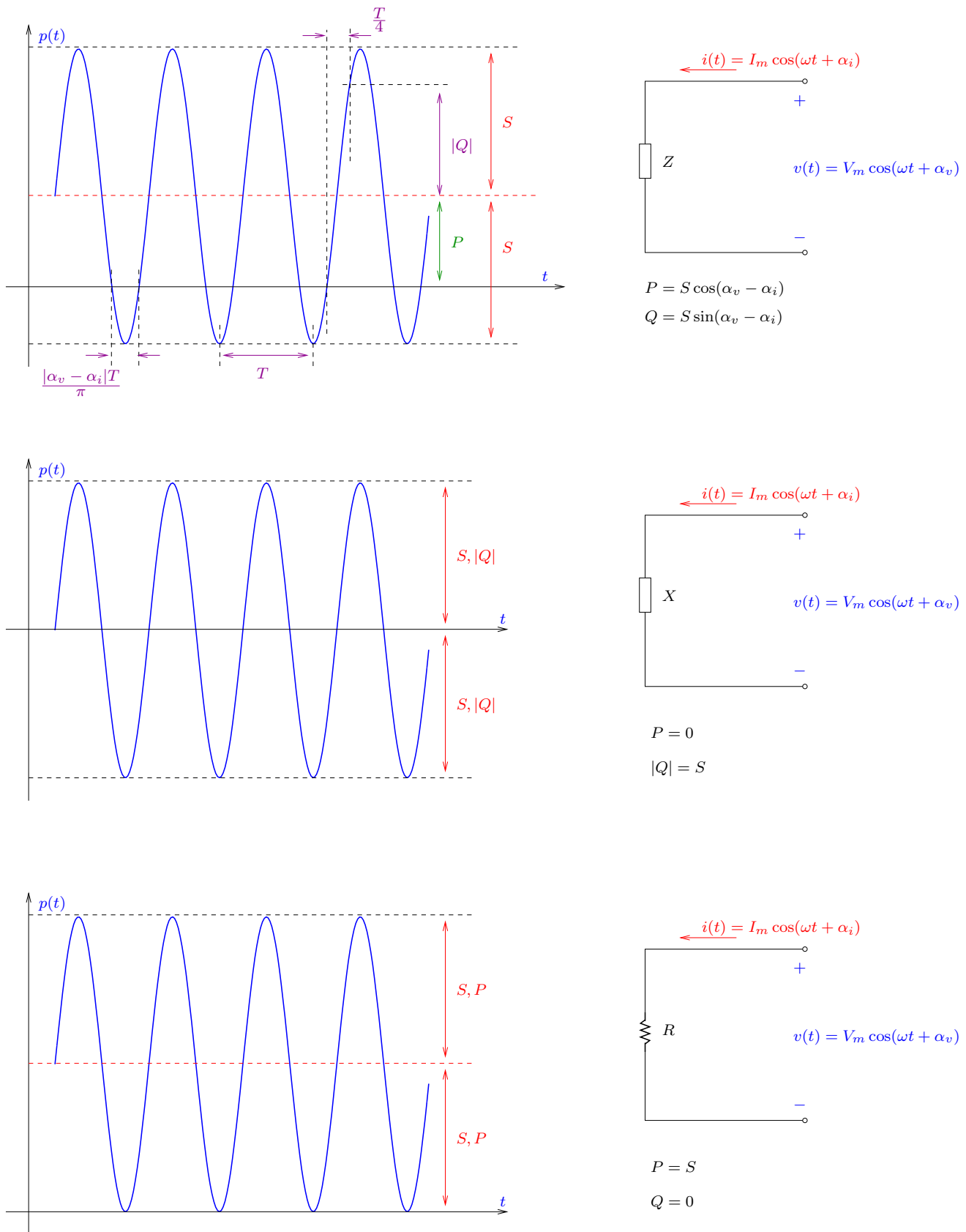


Figure 6: The relationship between the various types of power.

- The peak power absorbed by the load is $S + P$.
- The maximum rate at which energy is returned to the source is $S - P$. Note that $0 \leq S - P \leq |Q|$. While the maximum rate at which the reactive components return energy is $|Q|$, part of this energy is used by the resistive components, and thus not all of it is sent back to the source.

The Power Factor. The **power factor** is defined as $PF = \cos(\alpha_v - \alpha_i)$. When $PF = 1$, the load is equivalent to a resistor and energy flows always from the source to the load; the load does not return any energy back to the source. When $PF < 1$, in every cycle of $p(t)$, the load takes more energy than it needs and returns the excess of energy back to the source. Ideally, this would not be a problem. In practice, however, energy cannot be transferred without losses. Therefore, it is best if the load takes exactly as much energy as it needs, without returning any back to the source. For this reason, power factor correction is used to ensure that the load operates at a power factor $PF = 1$. The $p(t)$ curves with the same average power P but different power factors are illustrated in Figure 7.

In power factor correction, an additional reactive component is connected in parallel to the load so as to ensure that the overall power factor is 1. Three possible methods that could be used to calculate the value of the additional component are as follows.

- Method 1: Find the load current \mathbf{I} in the frequency domain. Let \mathbf{I}_Y be the component of \mathbf{I} perpendicular on the load voltage \mathbf{V} . Find a reactive component that has the current $-\mathbf{I}_Y$ when the load voltage \mathbf{V} is applied to it.
- Method 2: Find the admittance of the load in the form $\mathbf{Y} = G + jB$. Find a reactive component of admittance $-jB$.
- Method 3: Find the reactive power of the load Q . Find a reactive component that absorbs the reactive power $-Q$ when the load voltage is applied to it.

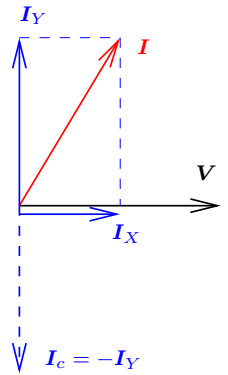


Table 1: Examples of power calculations.

	$P_R = I_{r,rms}^2 R = \frac{V_{r,rms}^2}{R}$	$Q_R = 0$
	$P_X = 0$	$Q_X = I_{x,rms}^2 X = \frac{V_{x,rms}^2}{X}$
	$P = P_{R_1} + P_{R_2}$	$Q = Q_{X_1} + Q_{X_2} + Q_{X_3}$

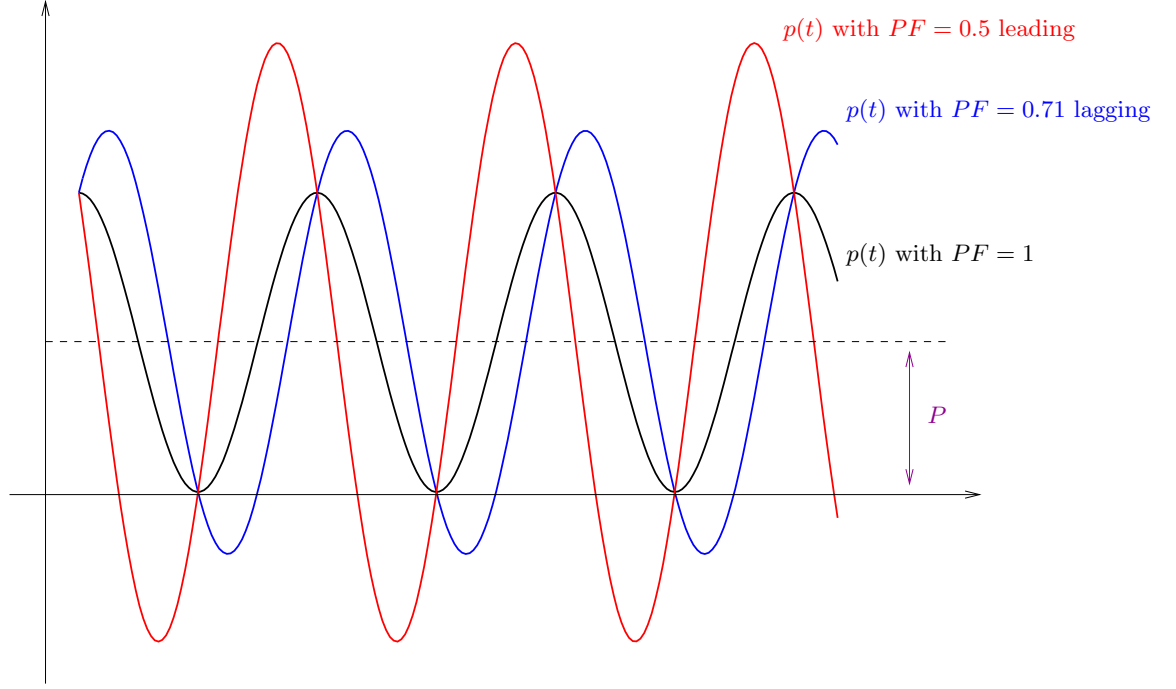


Figure 7: $p(t)$ curves with the same average power P but different power factors. The smallest curve has $PF = 1$ and is optimal.

The Complex Power. A convenient way to represent the apparent, average, and reactive powers is by means of the complex power. Consider an n -terminal component or network. Assuming the notation of Figure 1(b), let $\mathbf{V}_{k,rms}$ and $\mathbf{I}_{k,rms}$ denote the rms phasor representation of $v_k(t)$ and $i_k(t)$ for $k = 1, 2, \dots, n$. The complex power is

$$\mathbf{S} = \sum_{k=1}^n \mathbf{V}_{k,rms} \mathbf{I}_{k,rms}^* \quad (34)$$

For the two-terminal component of Figure 1(a), the equation simplifies to

$$\mathbf{S} = \mathbf{V}_{rms} \mathbf{I}_{rms}^* \quad (35)$$

where \mathbf{V}_{rms} and \mathbf{I}_{rms} denote the rms phasor representation of $v(t)$ and $i(t)$. Equation (34) can also be written in terms of the real and imaginary parts:

$$\mathbf{S} = \sum_{k=1}^n V_{k,rms} I_{k,rms} \cos(\alpha_{v,k} - \alpha_{i,k}) + j \sum_{k=1}^n V_{k,rms} I_{k,rms} \sin(\alpha_{v,k} - \alpha_{i,k}). \quad (36)$$

Let $\overline{f(t)}$ denote the average of a function $f(t)$. Using a derivation similar to that of equations (7) and (25), the complex power can be written as

$$\mathbf{S} = \sum_{k=1}^n \overline{v_k(t) i_k(t)} + j \sum_{k=1}^n \overline{v_k(t) i_k(t + t_0)}. \quad (37)$$

In view of equations (2) and (22), since $P = \overline{p(t)}$ and $Q = \overline{q(t)}$, we conclude that

$$\mathbf{S} = P + jQ \quad (38)$$

Consider a network consisting of m components. Let \mathbf{S}_k be the complex power of component k , for $k = 1, 2, \dots, m$. In view of equations (28) and (33), the complex power of the network will be

$$\mathbf{S} = \sum_{k=1}^m \mathbf{S}_k. \quad (39)$$